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Identical Particles and the Simplicial Category

G. V. Khatcherian[1,2]

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Combinatorial aspects of Bose and Fermi ensembles of "identical particles" are reflected in the simplicial category.

1. INTRODUCTION

With varying degrees of immediacy, both mathematics and physics emanate from, and are expressions of, our perception/conception of the world (Bohm, 1965, appendix: "Physics and Perception"). Thus, the "special relationship" between the two disciplines is simply a reflection of their underlying unity.

The evidence for unity lies not so much in the banal observation that the two subjects share vast tracts of formalism, but in that they share the intuitive and informal insights that give rise to these (Bohm, 1976). It is to be expected, therefore, conceptual novelties in the one discipline would have a direct bearing on the other.

Even though quantum theory has brought about drastic changes in our conception of the world, and, concomitantly, in the very foundations of physics, it has not had as pronounced an impact on the (conceptual) foundations of mathematics (not to be equated with the technical "foundations of mathematics," the main role of which is not to found, but to founder).

Our research in this direction takes root in set-theoretical considerations, motivated by the three distinct notions of "ensemble" that underly Maxwell–Boltzmann, Bose–Einstein, and Fermi–Dirac statistics. This aspect, viewed from a different angle, is the specific theme of the present paper.

[1]Department of Physics, Birkbeck College, University of London, London, England.
[2]Present address: 91 Hatherley Court, Hatherley Grove, London W2 5RD, United Kingdom.

2. ABSTRACT BOSE ENSEMBLES

Consider the notion of a "Bose ensemble" of "identical particles"; (Fermi ensembles will not be discussed till much later). Abstracting from the physical context, and thinking naively of the well-known combinatorial calculations (with stars and bars; as presented in, say, Feller, 1968, Chap. 2), a direct comparison with the notion of "set" suggests itself forcefully. For example, a Bose ensemble of two "identical particles" may be thought of—as regards the calculations—as a collection of two "indistinguishable elements", denoted suggestively by $\{*, *\}$, in contradistinction with $\{a, b\}$.

Accepting that set theory arises from the intuitive idea of collections of (distinguishable) elements, it seems plausible to ask whether the equally intuitive idea of collections of "indistinguishable elements," abstracted from the quantum domain, could serve as a clue for "quantum set theory," where some of the aspects hinted at above are inbuilt from the start.

Of course, we all know how to model the Bose situation in the usual framework, which, at the combinatorial level, amounts to working with sets "mod all permutations" (see Section 8); but here we are looking for a theory of (abstract) Bose ensembles that will stand on its own two feet.

In the present paper we show the question just raised—when suitably rephrased—has a surprisingly simple answer: the distinctive combinatorial features associated with Bose ensembles are already displayed in the simplicial category.

But this is to anticipate; first we must prepare the ground.

3. A CURSORY COMPARISON WITH TRADITIONAL SET THEORY

To begin with, we would like to discard the idea that somehow a theory of (abstract) Bose ensembles may be contrived along the lines of (any one version of membership-based) traditional axiomatic set theory. Let us note right away that the distinction between the usual and the Bose notions of "ensemble" does not involve—at least after abstraction—sophisticated metaphysical considerations. As our comparison above illustrates, it can be seen very plainly in terms of finite collections of two sorts of "elements."

On the other hand, one of the main concerns—if not the very *raison d'être*—of traditional axiomatic set theory has been to contend with the subtleties of transfinite sets. In subordination to this arduous task, or so it would seem to the outsider, finite sets have been adorned with otherwise not so compelling manipulations ("building up" from the empty set,...). Thus, the notion of set has lost its innocence in traditional set-theoretical thought; and the currently favored axiom systems (for example, Takeuti and Zaring, 1970) reflect a measure of estrangement from intuition, even where finite sets are concerned (see the criticisms of Lawvere, 1976, Section 2). Reason enough, for us, to discard this line of attack.

Still, it is useful to proceed a little further with our comparison, but without getting involved in technical details unwarranted by our requirements. Thus, let us recall Cantor's definition: "By a 'set' we mean any collection M into a whole of definite, distinct objects m (which are called the 'elements' of M) of our perception or our thought" (Dauben, 1979, p. 170). Comparing this "classical" notion with the Bose notion of ensemble, it is clear that the "elements" of the latter, though distinct, are not distinguishable. In a Bose ensemble—we are tempted to say—"all elements are equal"; or at least, they are "more equal" than their classical counterparts.

This is not merely a play on words; on the face of it, the concept of "identical particles" does suggest a reexamination of the notions of "identity" and "equality" (see, for example, Weyl, 1949). Yet, whereas physicists were startled by the novelty inherent in the calculations of Bose and Einstein (back in 1916–1917, in the context of blackbody radiation), most logicians concerned with set theory and "foundations" seem to have stayed aloof. Thus, to quote a recent source (Wang, 1974, p. 181), "the members [of a set] may be objects of any sort: plants, animals, photons, numbers, functions, sets, etc." Granting that in certain contexts we do regard photons as "definite objects" (in our calculations, to explain some aspects of our perceptions)—a view we ourselves have adopted to arrive at abstract Bose ensembles—we must nevertheless think of a collection of n pebbles and a collection of n photons in the same state as two different sorts of collections. Photons aggregate differently, and our mathematics ought to reflect this fundamental fact at a duly fundamental level.

Taking into account the above considerations (which will be further clarified in the rest of this paper) and seeking a ground level formalization of Bose ensembles (not out of pure love for formality, but hopefully to expose some essential aspects of the notion in the process) if we were to proceed by way of contemplating an axiomatization along the lines of traditional set theory, we would be led astray. Thinking naively, if we took ε as primary, it seems clear we will not get to the heart of the matter since both sorts of "elements" are quite comfortably members of their respective ensembles in a uniform manner—how would ε tell them apart? Or, are we to transfer the problem outside set theory proper, and devise formal languages with various notions of "equality"? It would appear a perplexing task, to say the least, to "dissect" the notion of equality.[3] Intuition would be sacrificed, and we are back to our original objection.

[3]Nevertheless, see Parker-Rhodes, 1981.

There is, however, an entirely different approach to set theory, and to our quest, which is much closer to our intuition, and we now turn our attention to it; [in defiance of Cantor's warning: "My theory stands as firm as a rock; every arrow directed against it will quickly return to its archer" (Dauben, 1979, p. 298)].

4. CATEGORICAL SET THEORY

The elementary theory of the category of sets (Lawvere, 1964) was the first formalization of set theory to depart radically from the traditional stance. Relegating elements and membership to a secondary role, and elevating mappings and composition to a primary one, the axioms described the category of sets, in all fidelity to our informal understanding and use of it in usual mathematical practice.

Crucial simplifying insights, and an eye for the assimilation of substantial, though seemingly unrelated, developments in geometry and logic into a unified conceptual framework, inspired Lawvere and Tierney to modify Lawvere's 1964 formulation to a much more versatile list of axioms, the elementary theory of topoi. Consequently, the importance of Lawvere's pioneering 1964 work is now viewed in a historical perspective: it extricated our thinking from the awesome hold of tradition, established the respectability of the "purified" notion of abstract sets (to be discussed), and, of course, helped pave the way for elementary topos theory.

Given a modicum of knowledge of the language and constructs of category theory, the modified axioms are deceptively easy to state: a topos is a cartesian closed category which has a subobject classifier (see Johnstone, 1977; Goldblatt, 1979; MacLane, 1975, gives a short introductory account). On one level, just like the 1964 precursor, the axioms express (succinctly, in terms of interrelated pairs of adjoint functors) some of our usual ideas about "the" category of sets and mappings; and are presented in this plausible manner by Lawvere (1976, 1973). But on a deeper level, they do much more: they provide an effective basis for the description and study of not only "the" category of (constant) sets (a topos among others), but also of more refined categories (topoi) of "continuously variable sets," which are now seen to pervade mathematics and logic (Lawvere, 1975, 1976; Tierney, 1972).

Elementary topos theory, or more suggestively, the theory of continuously variable sets, generalizes the very notion of (the category of constant) sets in an intuitive and conceptual manner, and—recalling our remarks in the introduction—will no doubt come to bear on theoretical physics (see Lawvere, 1980; Kock, 1981).

It is not within the scope of the present work to survey these prospects, or to discuss the conceptual advances incorporated in elementary topos theory. In fact, to deal with Bose ensembles, we shall diverge from set theory at the combinatorial level of (the category of) finite sets and mappings. Thus, in the furtherance of our quest we shall need no more than an informal acquaintance with the spirit of the categorical approach to set theory, and we shall avail ourselves of nothing beyond the simplest notions of category theory (readily accessible in Mac Lane, 1971).

Instead, we shall require a brief informal probe into the physical origins of the notion of ("pure," abstract) sets and mappings, and their basic role in mathematics, touching upon those aspects that are strictly relevant to our argument. Thus, the plain discussion that follows simply hopes to clarify and promote the way we shall be looking at things; it aspires to no more than that, and clearly has no pretensions to historical accuracy either.

5. THE CALCULATORY ROLE OF THE CATEGORY OF ABSTRACT SETS

The commonplace notion of a finite collection of like "things" seen, put (set), or imagined together (ensemble) is a basic one in our thinking, and is amply reflected in our everyday language.[4] The mathematical notion of a "pure," abstract set ultimately emanates from this primitive idea of a collection, by a process of abstraction (and extrapolation, to infinite sets, etc.).

A crucial step toward abstraction was implicitly taken (we conjecture) back in antiquity, when, probably in early trade or accounting, divers collections of like "things" were uniformly represented by, say, collections of (as many) pebbles—irrespective of the specific nature of the members of the various collections, and the sundry differences between the members within a given collection; since a further step of abstraction—from pebbles to "elements"—leads us to essentially the "pure" notion of (constant) abstract set: "An abstract set X has elements each of which has no internal structure whatsoever[5]; X has no internal structure except for the equality

[4]The collective nouns of our every day vocabulary rarely refer to arbitrary collections of irregular "things"; rather, they mostly denote uniform collections of like or similar "things." For example, pack (of wolves), flock (of sheep), crowd (of people) etc. [Incidentally, this suggests our common notion of a collection relates to that of a (finite) discrete category.]

[5]Thus, all the elements in an abstract set are exactly alike, completely uniform. Indeed, the imposition of absolute uniformity—brought about by ignoring all irrelevant dissimilarities—is part and parcel of the process of abstraction from actual "things" to abstract "elements."

and inequality of pairs of elements, and has no external properties save its cardinality; . . ." (Lawvere, 1976, p. 119).

Though some concrete primitive precursor or other of the "modern" concept of abstract set was at hand very early on, the notion of general mapping (as against one-to-one correspondence) was not—not even implicitly—since the need for it did not arise. In commercial accounting one needed to count, and the early calculations using "sets" of pebbles were limited to "arithmetic" manipulations; for example, adding and deducting pebbles, to mirror income and outgo. Consequently, it was the concept of number that evolved soon after by a further step of abstraction from sets (which in effect impeded the discovery of the notion of mappings); and "to calculate" came to signify "to compute" with numbers. But in Latin, "calculus" means "little stone" or "pebble," and accordingly "to calculate" would mean "to move or to manipulate pebbles"—in accordance with the requirements at hand.

The requirements of "modern" mathematics have, of course, brought into prominence general calculations with sets (and better still, variable sets) which find concrete expression in (or through) the notion of mapping—and composition. It would seem, however, some notion of mapping (without any idea of composition) was implicit in simple combinatorial calculations, underlying the computation of discrete probabilities, sometime before the "modern" era. To be sure, the practice was to speak of, say, balls and cells, and to ask for the number of ways of placing m balls in n cells; but this must have entailed the calculation of all mappings between the given collections.

The fundamental importance of abstract sets to mathematics rests in that they provide the means to contemplate mappings, and, thereby, to calculate. "The only possible use of abstract sets T is the possibility of indexing or parametrizing things by the elements of T in the hope of clarifying actual relations between the things by means of calculations on mappings introduced to mirror the relations . . ." (Lawvere, 1976, p. 120, 121). The categorical approach to set theory gives immediate prominence to the mappings, the basic (ground level) activity of calculating, which, after all, is the carrier of mathematical meaning. In primitive vernacular the palpability of this observation comes out in trenchant fashion: mathematical meaning is displayed by the active movements—or, if you prefer, the mental assignments—that the pebbles undergo; not the dead lumps of stone in themselves.

Now, following one basic step of calculation (i.e., mapping) by another, we conceive composition, and—to cut it short—"the" category of abstract sets and mappings $\mathscr{S}$; or, going strictly by our mundane argument, the category of finite abstract sets and mappings $\square$.

Using $\mathscr{S}$ as a base, we set out, and with more finesse, we set up (using standard methods of category/topos theory) more complex calculations; in the hope of obtaining "concrete" representations of more refined (less abstract) conceptual categories, that arise in mathematical practice, in the course of our ongoing efforts to render explicit various aspects of our perception/conception of the world.

Thus, we wish to promote the view (to be clarified further in the sequel) that the category $\mathscr{S}$—and with obvious limitations, $\square$—serves as a rudimentary "abstract space," where we perform, or imagine to perform, our ground level, "concrete" calculations.

6. A CATEGORY OF BOSE ENSEMBLES?

Reviewing our comparison of Bose ensembles and classical ensembles ($\approx$finite abstract sets), it is by now fairly clear that our earlier discussion in Sections 1 and 2—suggesting a description of $\{*, *\}$ versus $\{a, b\}$ at face value—was somewhat superficial, since it neglected the essential question: how do we thereby calculate? Bringing in this essential component, in the classical case we arrive at the category $\square$ of finite sets and mappings. The specific aim of this paper, accordingly rephrased, is to seek out the "parallel" Bose category $\mathbb{B}$.

Thus, we must look for morphisms and composition; and in this quest we are guided by the well-known calculations with stars and bars (Feller; or any introductory text on quantum physics) used to compute the correct probabilities for Bose-Einstein statistics. But alas, it is not even clear what the objects of this projected Bose category $\mathbb{B}$ are. As domain, they are apparently Bose ensembles (with "elements" the indistinguishable stars); while as codomain, they are apparently classical ensembles (with "elements" the distinguishable cells, between the bars)—and the trouble is the objects must fulfil both roles.

This, however, is only an apparent impasse; having to do with the standard manner the subject of Bose ensembles is presented, and to some extent, the informal intuitive explanation, "identical particles", the stars are meant to suggest. To see this, we must pay a little more attention to the physical origins of $\square$ itself.

7. CLASSICAL PARTICLES AND THE CATEGORY OF FINITE SETS

From our earlier discussion—the simile of pebbles and elements—it is clear that the category of finite sets $\square$ is abstracted from what we would call a classical domain of experience, where by "classical" we fully intend the physical connotations.

More emphatically, motivated by Bohm's views on physics and perception (Bohm, 1965, appendix), we claim that very rudimentary aspects of the motion of classical particles ($\approx$ idealized pebbles $\approx$ elements $\approx$ points) in ordinary space are also reflected in □, through the mappings, in the following sense.

Just as the notion of "elements" arises from the idea of "pebbles" in its most abstract form, by shedding considerations of size, mass, distinctive features, etc., but retaining some idea of their individual positions; and just as the notion of (finite) set arises from the idea of their "togetherness" in its most abstract form, by shedding considerations of topology, structure, etc., but retaining some idea of their being in the same location; so the notion of mapping emanates from the idea of their collective motion or transportation in space, by shedding considerations of continuity, dynamics, etc., but still retaining, in abstract form, vestiges of their individual trajectories. (After all, in mathematical practice we often do talk of a mapping taking this element to that element, or even, of this element going to that.)

Though this spatial view of □ deserves further elaboration, we feel it is best to illustrate what we are driving at by means of a simple example, which will also serve to clarify aspects of Bose ensembles and statistics. Let us consider a fast revolving target disk divided into six equal sections, labelled 0, 1, . . . , 5, and two air guns aimed to hit it. We fire the two guns, staggering the shots a little to allow the disk to revolve, so as to get "random" or haphazard hits. (We could consider the throwing of two dice, but our example will bring out the points we wish to illustrate more directly.)

We envisage 36 "basic possibilities," and on the assumption that they are equally probable, we arrive at the probability of a specified outcome by counting the number of ways in which it may come about, and dividing by the total number of basic possibilities. Thus, the probability of obtaining a double 5 equals 1/36; while the probability of obtaining a 4 and a 5 equals 2/36.

A crude assessment of the physical situation, but nevertheless a fundamental one; more detailed consideration may prompt us to modify the assumption of "equally probable," but we are very relucant to relinquish the assumption of 36 basic possibilities. How does this number dawn on us?

A simple and intuitive way to view the above experimental setup—in abstract form, idealized to the bare essentials for the purposes of working out the probabilities—is to represent the two guns, or nozzles, by the elements of a set 2, and the six sections of the target by the elements of a set 6. Each trial of the experiment is then represented by a mapping $2 \to 6$. We can, of course, exhibit such a mapping by drawing a little diagram showing the two sets with their elements, and the assignments making up the mapping, explicitly. We suggest that this, in turn, is a rudimentary pictorial representation of the experiment, showing the separate trajectories of the two pellets, in idealized fashion (i.e., mod details which are judged irrelevant for the calculations intended). We are now in a position to carry out our calculations; we note that there are 6^2 ($\approx 6^{(1+1)} \approx 6^1 \times 6^1 \approx 6 \times 6 \approx 36$) different mappings from 2 to 6—standing for the basic possibilities—and that only one of these will bring about the outcome double 5, while two of these will bring about the outcome 4 and 5; whence our assessment of the corresponding probabilities.

The point of detailing these banalities is to remind ourselves that the above considerations, leading to 36 basic possibilities, amount to a theory—albeit a very primitive one. In essence, this theory is based on an insight arising from our intuitive perception of the experiment: that the given experiment is similar, in certain relevant aspects, to a more familiar experiment; namely, that of "calculating" mappings from a collection of two pebbles to a collection of six pebbles.

Now, we would like to emphasize that the question of whether the pellets (and, for the matter, the representing pebbles) are identical or not is not a crucial factor in the experimental arrangements. For all we care, the pellets—taken from one manufacturer's box—could be all exactly alike, identical to the hilt. The possibility of actually (i.e., physically) labeling the pellets, is incidental to the workings of the experiment. Of course, it could be argued that in principle we could mark the pellets with little scratches, to distinguish between the two; but say the pellets melt on impact with the target. Right, we could arrange for the pellets to be cast in two different alloys; but say the pellets go clean through the target, leaving two little holes, which is quite enough to determine the outcomes we are interested in. Whatever these incidental arrangements, we would still work out our probabilities in the same way. In other words, if we actually use pellets similar in every detail, the experimental results are not affected.

The crucial observation is that the given experiment involves two "classical particles" which have an autonomous existence throughout their independent flight to the target, in "classical" space. And while in our primitive theory we have not made detailed stipulations concerning the experimental setup, or "classical" space, we have—even at the very abstract level of considering just mappings from 2 to 6—included enough features to enable us to arrive at the "correct" probabilities.

8. IDENTICAL PARTICLES AND THE SIMPLICIAL CATEGORY

Instead of classical particles, let us now consider "identical particles" satisfying Bose–Einstein statistics. For the sake of comparison with the previous example, consider a hypothetical "experiment," with two "sources," and six target "states", labeled $0, 1, \ldots, 5$. Two identical bosons—one from each source—will end up in the various states available; (it would not be amiss to think roughly on the lines of the informal discussion in Feynman, 1965, Section 4-2; but here we wish to proceed in a combinatorial manner, talking of "basic possibilities," rather than "probability amplitudes," etc.).

According to the well-known combinatorial calculations (Feller), the probability for both bosons to get into state 5 equals 1/21; the probability for the bosons to get into states 4 and 5 equals, again, 1/21. A situation drastically different from the classical: 21 basic possibilities, as against 36, and only one of these brings about the outcome 4 and 5.

Going by the usually offered informal explanations—which are as much part of a theory as the formal calculations—the standard assessment of the difference between the classical and the Bose situations is indicated by the slogan "identical particles"; whence the star notation $\{*, *\}$ for the bosons (which stood in our way in Section 5). "The problem of identity which we are facing here has for thousands of years been one of the most baffling in metaphysical speculations" (Jauch, 1968, Section 15-3), which quantum theory seems to have "brought down from a purely speculative level to the empirical level." We are asked, it would seem (for instance, Jauch, 1968, p. 276; Weyl, 1949), to contemplate a profound difference between two particles that are merely similar but not "identical," and two particles that are "identical," and not merely similar in every respect. We suggest that the informal explanation "identical particles" is a convenient reminder of the standard maneuvres (of symmetrization in the Bose case, and antisymmetrization in the Fermi case) to be undertaken to ultimately get hold of the correct (amplitudes, hence) probabilities; but otherwise it is not all that elucidating.

As in the classical situation, let us start off by representing the two sources by the elements of a set 2, and the six target states by the elements of a set 6. The "Bose experiment," i.e., the process of two bosons getting from the sources to the states, is then represented schematically by a morphism or arrow $2 \to 6$, which must be defined in such a way as to get 21 arrows from 2 to 6, as against 36—so the arrows do not stand for arbitrary mappings, which ought to be clear on physical grounds in any case. In our quest for an explicit definition of these arrows, let us be guided by our intuitive understanding of the difference between the two physical situations (also touched upon by Jauch, 1968, p. 276, second and third paragraphs). In the Bose case, we talk of two "particles" mainly in a metaphorical sense; in the process of getting from the sources to the states, these "particles" do not have an autonomous existence, independent of each other; nor, of course, explicit trajectories in "classical" space. At the rarified level of the very rudimentary mathematics we have intentionally restricted ourselves, we must somewhow express, at least, the interinvolvement or interdependence of the two bosons, during their progress from the sources to the states.

The way to give mathematical expression, at the ground level, to this idea that the bosons proceed "from 2 to 6" in a more interdependent or orderly manner than that expressed by an arbitrary mapping, is to consider the sets 2 and 6 as linearly ordered, and to ask for $2 \to 6$ to be an order-preserving mapping: (as shown in the appendix) there are precisely 21 such morphisms from 2 to 6—as required—which can be taken to stand for the basic possibilities in our calculations relating to the Bose "experiment."

This suggests that as far as our calculations are concerned the simplicial category $\mathbb{\Delta}$ (see Appendix) relates to the Bose situation in much the same way as the category of finite sets $\square$ relates to the classical situation.

To see how our approach compares with the standard view of things, let us look a little more closely at the usual explanation in terms of "identical particles," and what it leads to. (In what follows we use m and n instead of 2 and 6, and we take for granted our notations in the Appendix.) Referring to the Bose situation, we would say that the m bosons cannot be individually named or distinguished; by which we in effect mean they are completely interchangeable in our calculations. Hence, if we are to work with a set m, we must insure "invariance under all permutations on the set m." In other words, to arrive at $\mathbb{B}(m, n)$—that is, the collection of "basic possibilities" in the Bose situation [as against $\square(m, n)$ in the classical]—we would start off with $\square(m, n)$ and let the symmetric group S_m of permutations on the domain m act on $\square(m, n)$, on the left, in the obvious manner.[6] We would then propose that the orbit set $\square(m, n)/S_m$ be taken as $\mathbb{B}(m, n)$. After all, this is the underlying combinatorial summary of the initial steps we go through when symmetrizing in the linear situation (see, for example, Jauch, 1968, Sections 15-4, 15-5).

These maneuvres do, of course, produce correct results; but from our combinatorial point of view, they are arrived at via a diversion. For we assert that there is a bijection $\square(m, n)/S_m \cong \mathbb{\Delta}(m, n)$ between the collection of orbits on the left, and the collection of morphisms on the right; namely: if (f) denotes the orbit of mapping f in $\square(m, n)$, under the action of S_m, there is precisely one f' in (f) which preserves order,[7] and the bijection is given by $(f) \mapsto f'$. Thus, at the combinatorial level, when we start off with the "classical" $\square(m, n)$ and go through the usual procedure of identifications brought about by symmetrization, we arrive at what amounts to $\mathbb{\Delta}(m, n)$.

[6]The action $S_m \times \square(m, n) \to \square(m, n)$ is given by $(\sigma, f) \mapsto \sigma \cdot f$, where $\sigma \cdot f$ denotes the composite of permutation σ followed by mapping f.

[7]We have implicitly endowed sets m and n with linear order; in this regard see our comments in the Appendix.

As a result of this diversion, in Section 6 we were at a loss as to what the objects of the projected Bose category were. Moreover, the obvious compositions

$$\Delta(m, n) \times \Delta(n, p) \to \Delta(m, p)$$

have been overlooked; since it takes some effort to see in terms of

$$\square(m, n)/S_m \times \square(n, p)/S_n \to \square(m, p)/S_m$$

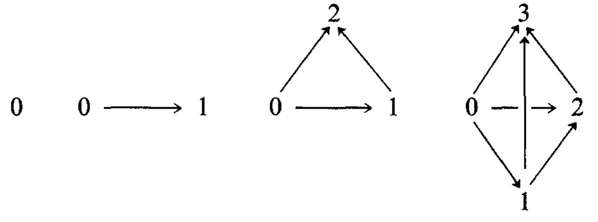
Thus, the category of abstract Bose ensembles $\mathbb{B}$, which eluded us in Section 6, turns out to be nothing but the simplicial category Δ. Here, rudimentary, yet basic, combinatorial features underlying Bose–Einstein statistics are inbuilt from the start. Furthermore, primitive aspects of "identical particles" satisfying Fermi–Dirac statistics can also be discussed in terms of Δ—to be precise, in terms of the monomorphisms in Δ[8]. Thus, it appears, a small fragment of the combinatorial, discrete facet of quantum theory is captured in Δ.

In view of the protean aspects of the simplicial category (summarized in MacLane, 1971, p. 176), the above observation suggests new lines of research, which will be discussed elsewhere.

APPENDIX

First we recall some familiar definitions, in order to establish notation. We define the simplicial category Δ (in accordance with MacLane, 1971, Chap. 7) as follows. We take as objects of Δ all finite linearly ordered sets, including the empty or initial 0. We denote the objects of Δ by $0, 1, 2, \ldots, m$, etc.; where m stands for the linearly ordered set $0 \leq 1 \leq 2 \leq \cdots \leq m-1$ with m elements. Though we use the same typescript for the objects of Δ and the elements within the various objects of Δ, they are quite distinct and must not be confused; (to boot, no nestings is implied by our notational abuse). The morphisms of Δ are defined as the order-preserving mappings $m \xrightarrow{f} n$ ($i \leq j$ in m implies $if \leq jf$ in n).

Alternatively, we could view each finite linearly ordered set as itself a category (using $\to$ instead of $\leq$) as follows:



1 2 3 4 and so on

[8]The monomorphisms in Δ are simply the order-preserving injective mappings.

in which case each order-preserving mapping $m \to n$ would be referred to as as functor, from category m to category n, and $\mathbb{A}$ would be described as the category of these finite categories—which is just another (better) way of putting the first definition.

The above diagrams are also suggestive of the name "simplicial," and yet another familiar presentation of $\mathbb{A}$, as the category (with objects the finite ordinals) generated by the "face" and "degeneracy" arrows

$$\delta_i^n : n \to n+1 \quad (i = 0, \ldots, n), \qquad \sigma_j^n : n+1 \to n \quad (j = 0, \ldots, n-1)$$

subject to the usual relations (see MacLane, 1971, p. 173).

The category $\mathbb{O}$ has as objects all finite (abstract) sets; and as morphisms, all mappings $m \xrightarrow{f} n$. The objects of $\mathbb{O}$ are also denoted by 0 (empty set), $1, 2, \ldots, m$, etc.; where now m stands for the set $\{0, 1, 2, \ldots, m-1\}$ with m elements. Again, the elements of m are here denoted by $0, 1, \ldots,$ purely as a matter of notational convenience—we definitely do not mean to suggest that our sets are nested within each other, or anything like that. (Any suggestion of this kind is, of course, dismissed by the very definition of an abstract set.)

On the other hand, $m = \{0, 1, 2, \ldots, m-1\}$ does suggest an implicit linear order on m—which brings us to the point raised in footnote 7, regarding the bijection $\mathbb{O}(m, n)/S_m \cong \mathbb{A}(m, n)$ (in Section 8, toward the end). When specifying this bijection, we are, strictly speaking, thinking not quite in terms of $\mathbb{O}$, but in terms of category "$\mathbb{O}$" defined as follows. The objects of "$\mathbb{O}$" are taken to be all finite linearly ordered sets (as in $\mathbb{A}$), but the morphisms of "$\mathbb{O}$" are still defined to be arbitrary mappings, as in $\mathbb{O}$. The categories "$\mathbb{O}$" and $\mathbb{O}$ are clearly equivalent, and we choose to gloss over the distinction between the two. Now, corresponding to the well-known

$$|\mathbb{O}(m, n)| = n^m$$

[where $\mathbb{O}(m, n)$ denotes, as usual, the collection of all morphisms from object m to object n in category $\mathbb{O}$] we have

$$|\mathbb{A}(m, n)| = (m+n-1)!/m!(n-1)!$$

[in particular, $|\mathbb{A}(0, n)| = 1$ for every n in $\mathbb{A}$; $|\mathbb{A}(m, 0)| = 0$ if $m \neq 0$; and $|\mathbb{A}(2, 6)| = 21$, as asserted in Section 8], which follows from the observation that what goes under the description of distinguishable distribution of m indistinguishable balls in n cells (Feller, 1968, Chapt. 2, Section 5) can be viewed simply as a morphism $m \to n$ in $\mathbb{A}$.

To see this, recall the usual manner in which we work out the number

$$|\mathbb{B}(m, n)| = (m+n-1)!/m!(n-1)!$$

of all "distinguishable distributions of m indistinguishable balls in n cells," in the context of Bose–Einstein statistics. For example,

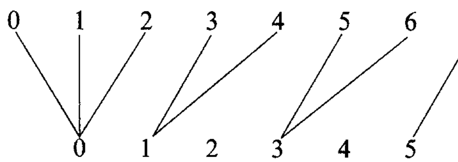
$$/ * * * / * * / \quad / * * / \quad / * /$$

is one possible "distribution," with $m=8$ and $n=6$, where we use seven bars to represent the six cells by the spaces in between.

Keeping the two outer bars fixed, we note that each shuffle[9] of the m stars and the $(n-1)$ inner bars amounts to a "distinguishable distribution" in $\mathbb{B}(m, n)$—whence the number $(m+n-1)!/m!(n-1)!$.

We now indicate a bijection $\mathbb{B}(m, n) \cong \mathbb{\Delta}(m, n)$.

Under this bijection, the distribution in $\mathbb{B}(8, 6)$ given above corresponds to the morphism



in $\mathbb{\Delta}(8, 6)$

Here we have introduced an explicit domain 8 and an explicit codomain 6, whose elements represent the "cells." We wish to think of each distribution as the end result of some mapping from 8 to 6; the positions of the stars giving some indication as to what assignments have taken place. But the requirement that the balls or stars be "indistinguishable" in effect discounts precisely those mappings from 8 to 6 which do not preserve order. In other words, a mapping f: $8 \to 6$ which does not preserve order cannot be distinguished from the (unique) order-preserving mapping f': $8 \to 6$ which hits the same elements in 6, the same number of times, as f. For, in this context, the requirement that the stars be considered "indistinguishable" translates to the instruction that for each mapping f: $8 \to 6$ we are to ignore which particular element of the domain is assigned to a given "cell" in the codomain. So we might as well suppose that the first three stars (in cell 0, at the extreme left) come from the first three elements $(0, 1, 2)$ in 8; the next two stars (in cell 1) from the elements 3, 4 in 8, and so on. Thus we obtain an order preserving mapping $8 \to 6$, as shown, which conveys the same information as the distribution

$$/ * * * / * * / \quad / * * / \quad / * /$$

This informal argument allows us to view each distribution in $\mathbb{B}(m, n)$ as an order-preserving mapping—i.e., a morphism in $\mathbb{\Delta}(m, n)$—and the other

[9]Our use of "shuffle" is in agreement with MacLane's definition (see Mac Lane, 1975b, p. 243).

way around. Hence the bijection $\mathbb{B}(m, n) \cong \mathbb{\Delta}(m, n)$ and $|\mathbb{\Delta}(m, n)| = (m+n-1)!/m!(n-1)!$. (For a more formal specification of this bijection, see Khatcherian, 1983, Chaps. 14, 15.)

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REFERENCES

Bohm, D. (1965). *The Special Theory of Relativity*. Benjamin, Reading, Massachusetts.

Bohm, D. (1976). On the creation of a deeper insight into what may underlie quantum physical law, in *Quantum Mechanics, Determinism, Causality and Particles*, M. Fano *et al.*, eds., pp. 1-10. Reidel, Dordrecht.

Dauben, J. W. (1979). *Georg Cantor—His Mathematics and Philosophy of the Infinite*. Harvard Univerity Press, Cambridge, Massachusetts.

Feller, W. (1968). *An Introduction to Probability theory and its Applications*, Vol. I (third ed.). John Wiley, New York.

Feynmann, R. P. (1965). *The Feynmann Lectures on Physics*, Vol III. Addison-Wesley, Reading, Massachusetts.

Goldblatt, R. (1979). *Topoi—The Categorical Analysis of Logic*. North-Holland, Amsterdam.

Jauch, J. M. (1968). *Foundations of Quantum Mechanics*. Addison-Wesley, Reading, Massachusetts.

Johnstone, P. (1977). *Topos Theory. LMS Monograph 10*. Academic Press, New York.

Khatcherian, G. (1983). Ph.D. thesis. Birkbeck College, University of London.

Kock, A. (1981). *Synthetic Differential Geometry. LMS Lecture Notes 51*. Cambridge University Press, Cambridge.

Lawvere, F. W. (1964). An elementary theory of the category of sets, *Proceedings of the National Academy of Sciences, USA*, **52**, 1506-1511.

Lawvere, F. W. (1973). Theoria della categorie sopra un topos di base. Lectures delivered at Perugia University, 1971-1973, unpublished.

Lawvere, F. W. (1975). Continuously variable sets, algebraic geometry = geometric logic, in *Logic Colloquium '73*, H. E. Rose and J. C. Shepherdson, eds., pp. 135-156. North-Holland, Amsterdam.

Lawvere, F. W. (1976). Variable quantities and variable structures in topoi, in *Algebra, Topology, and Category Theory*, a collection of papers in honor of Samuel Eilenberg, A. Heller and M. Tierney, eds., pp. 101-131. Academic Press, New York.

Lawvere, F. W. (1980). Toward the description in a smooth topos of the dynamically possible motions and deformations of a continuous body, *Cah. Topol. Geometr. Different.* Vol. XXI-4, 377-392.

MacLane, S. (1971). *Categories for the Working Mathematician*, GTM Vol 5. Springer-Verlag, Berlin.

MacLane, S. (1975). Sets, topoi, and internal logic in categories, in *Logic Colloquium '73*, H. E. Rose and J. C. Shepherdson, eds., pp. 119-133. North-Holland, Amsterdam.

MacLane, S. (1975b). *Homology* (third corrected printing). Springer-Verlag, Berlin.

Parker-Rhodes, A. F. (1981). *The Theory of Indistinguishables.* Reidel, Dordrecht.
Takeuti, G., and Zaring, W. M. (1971). *Introduction to Axiomatic Set Theory,* GTM Vol. 1. Springer-Verlag, Berlin.
Tierney, M. (1972). Sheaf theory and the continuum hypothesis, in *Toposes, Algebraic Geometry and Logic.* Springer LNM Vol. 274, F. W. Lawvere, ed., pp. 13-42. Springer-Verlag, Berlin.
Wang, H. (1974). *From Philosophy to Mathematics.* Routledge and Kegan Paul, London.
Weyl, H. (1949). *Philosophy of Mathematics and Natural Science.* Princeton University Press, Princeton, New Jersey.